# The Noncommutative Edmonds' Problem Re-visited 

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#### Abstract

Let $T$ be a matrix whose entries are linear forms over the noncommutative variables $x_{1}, x_{2}, \ldots, x_{n}$. The noncommutative Edmonds' problem (NSINGULAR) aims to determine whether $T$ is invertible in the free skew field generated by $x_{1}, x_{2}, \ldots, x_{n}$. Currently, there are three different deterministic polynomial-time algorithms to solve this problem: using operator scaling [1], algebraic methods [2], and convex optimization [3].

In this paper, we present a simpler algorithm for the NSINGULAR problem. While our algorithmic template is similar to the one in [2], it significantly differs in its implementation of the rank increment step. Instead of computing the limit of a second Wong sequence, we reduce the problem to the polynomial identity testing (PIT) of noncommutative algebraic branching programs (ABPs).

This enables us to bound the bit-complexity of the algorithm over $\mathbb{Q}$ without requiring special care. Moreover, the rank increment step can be implemented in quasipolynomial-time even without an explicit description of the coefficient matrices in $T$. This is possible by exploiting the connection with the black-box PIT of noncommutative ABPs [4].


[^0]
## 1 Introduction

Let $\underline{x}=\left\{x_{1}, \ldots, x_{n}\right\}$ be $n$ variables and $\mathbb{F}$ be a field. Consider the coefficient matrices $A_{0}, A_{1}, \ldots, A_{n} \in \mathbf{M}_{s}(\mathbb{F})$, and define an $s \times s$ symbolic matrix $T$ as $T=A_{0}+A_{1} x_{1}+\ldots+A_{n} x_{n}$. In 1967, Edmonds introduced the problem of deciding whether $T$ is invertible over the rational function field $\mathbb{F}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ [5], often referred to as the SINGULAR problem. More generally, Edmonds was interested in computing the (commutative) rank of $T$ over the rational function field $\mathbb{F}\left(x_{1}, \ldots, x_{n}\right)$. Equivalently, the problem asks to compute the maximum rank of a matrix in the matrix space generated by the F -linear span of the coefficient matrices. The problem was further studied by Lovász [6], in the context of graph matching and matroid-related problems. The SINGULAR problem, more generally the rank computation problem, admits a simple randomized polynomial-time algorithm due to the Polynomial Identity Lemma [7, 8, 9]. However, the quest for an efficient deterministic algorithm remains elusive. Eventually, Kabanets and Impagliazzo showed that any efficient deterministic algorithm for SINGULAR would lead to a strong circuit lower bound, justifying the elusiveness over the years [10]. Interestingly, the commutative rank computation problem admits a deterministic PTAS algorithm [11].

The rank computation problem is also well-studied in the noncommutative setting [12, 13]. More precisely, $T$ is still a linear matrix but the variables $x_{1}, \ldots, x_{n}$ are noncommuting. The problem of testing whether $T$ is invertible (NSINGULAR), or the rank computation question is naturally addressed over the noncommutative analog of the commutative function field, the free skew field $\mathbb{F} \nless \underline{x}\rangle=\mathbb{F} \nless x_{1}, x_{2}, \ldots, x_{n} \gg$. The free skew field has been extensively studied in mathematics [14, 15, 16] and the definition is somewhat involved. However, for the purpose of this paper, it is enough to state that $\mathbb{F} \nless \underline{x}\rangle$ is the smallest field over the variables $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ when we drop even the commutativity relations. Similar to the commutative setting, the noncommutative rank computation of $T$ has an equivalent definition involving the matrix space generated by the F-linear span of the coefficient matrices. We say that the noncommutative rank of $T$ is $s-c$ if $c$ is the maximum integer such that there exists a $c$-shrunk subspac ${ }^{11}$ of the said matrix space.

Two independent breakthrough results showed that NSINGULAR $\in P$ [1, 2]. In particular, the algorithm of Garg, Gurvits, Oliveira, and Wigderson [1] is analytic in nature and based on operator scaling which works over $\mathbb{Q}$. The algorithm of Ivanyos, Qiao, and Subrahmanyam [2] is purely algebraic. Moreover, the algorithm in their paper [2] works over $\mathbb{Q}$ and fields with positive characteristics. Subsequently, a third algorithm based on convex optimization is also developed by Hamada and Hirai [3]. Not only these are beautiful results, but also they enriched the field of computational invariant theory greatly [17, 18].

In this paper, we propose a simpler algorithm for NSINGULAR. The main algorithmic template of our algorithm is similar to [2]. However, there is an important difference in implementing one of the core steps that we explain in the next subsection.

Theorem 1. Given an $s \times s$ matrix $T$ whose entries are $\mathbb{Q}$-linear forms over the noncommuting variables $\left\{x_{1}, \ldots, x_{n}\right\}$, the noncommutative rank of $T$ over $\mathbb{Q} \nless x_{1}, \ldots, x_{n} \gg$ can be computed in deterministic poly $(s, n)$ time. As a special case, NSINGULAR $\in P$.

Remark 2. The result of [2] works over fields of positive characteristics using the rudiments of Galois theory. Similarly, our algorithm can also be extended over fields of positive characteristics. Since the key algorithmic ideas remain exactly the same, we prefer to describe the algorithm only over $\mathbb{Q}$ to minimize the use of the Galois theory machinery.

[^1]
### 1.1 The Overview

The noncommutative rank (ncrank) of an $s \times s$ matrix $T$ over the free skew field is the minimum $r$ such that $T$ can be written as $T=P Q$ where $P$ and $Q$ are $s \times r$ and $r \times s$ matrices with linear entries. This is also referred to as the inner rank of $T$ [12]. There are several equivalent definitions of noncommutative rank (see [19, 2] for more details). The definition of our particular interest in this paper is the blow-up definition [2, 20]. Let $T$ be written as $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$ where $A_{0}, A_{1}, \ldots, A_{n}$ are the coefficient matrices. For any matrix tuple $\underline{p}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ of dimension $d$, the evaluation is defined as the following $s d \times s d$ matrix:

$$
T(\underline{p})=A_{0} \otimes I_{d}+\sum_{i=1}^{n} A_{i} \otimes p_{i} .
$$

Define $T^{\{d\}}$ as the set of $s d \times s d$ matrices such that for each $B \in T^{\{d\}}$, there is a $d$-dimensional matrix tuple $\underline{p}$ such that $B=T(\underline{p})$. Let $\operatorname{rank}\left(T^{\{d\}}\right)$ be the maximum rank attained by a matrix in $T^{\{d\}}$. The regularity lemma [2, 20] shows that $\operatorname{rank}\left(T^{\{d\}}\right)$ is a multiple of $d$. The blow-up definition of the noncommutative rank is the limit of the sequence $\left(\frac{\operatorname{rank}(T(\underline{p}))}{d}\right)$ as $d \rightarrow \infty$. It follows from [13, 2] that

$$
\operatorname{ncrank}(T)=\lim _{d \rightarrow \infty}\left(\frac{\operatorname{rank}(T(\underline{p}))}{d}\right) .
$$

In this paper, we present a deterministic polynomial-time algorithm to compute the noncommutative rank of a linear matrix. The algorithmic template is similar to the algorithm of Ivanyos, Qiao, and Subrahmanyam [2]. Before we present our algorithm, we first go over the key steps of the algorithm presented in [2]. Given a linear matrix $T$ of size $s$, their algorithm gradually computes the rank of $T$. Suppose the algorithm outputs a matrix in $T^{\{d\}}$ of rank at least $r d$ at any intermediate stage (where it is always maintained that $d \leqslant s+1$ ). The next stage consists of two main steps: (a) rank increment step, (b) rounding and blow-up control.

## (a) Rank Increment Step

Given a matrix $B$ in $T^{\{d\}}$ of rank $\geqslant r d$, the algorithm first checks whether there exists another matrix $B^{\prime}$ in $T^{\left\{d^{\prime}\right\}}$ (where $d^{\prime}>d$ ) of rank $\geqslant r d^{\prime}+1$. If no such matrix exists, we output ncrank $(T)=r$. The correctness follows from the blow-up definition. Otherwise, the algorithm finds such a $B^{\prime}$ and proceeds to the next step. This step is technically involved and uses a linear algebraic procedure involving the limit point computation of a second Wong sequence [21,2]. Informally speaking, this is analogous to the augmenting-path algorithm for bipartite matching. A direct implementation of the limit point computation incurs a bit-complexity blow-up over $\mathbb{Q}$ due to the repeated application of Gaussian eliminations. To tackle this, the notion of matrix pseudo-inverse is used [21, 2].

## (b) Rounding and Blow-up Control

Once we have the matrix $B^{\prime}$ in $T^{\left\{d^{\prime}\right\}}$ of rank $\geqslant r d^{\prime}+1$, the rounding step of the algorithm uses a constructive version of the regularity lemma to find another matrix $B^{\prime \prime}$ in $T^{\left\{d^{\prime}\right\}}$ such that the rank of $B^{\prime \prime}$ is a multiple of $d^{\prime}$ and $\operatorname{rank}\left(B^{\prime \prime}\right) \geqslant \operatorname{rank}\left(B^{\prime}\right)$. That implies the rank of $B^{\prime \prime}$ is $r^{\prime} d^{\prime}$ where $r^{\prime}$ is at least $r+1$. Here, $d^{\prime}$ is at least a constant multiple of $d$. One can not afford such a blow-up in the dimension during every round of the algorithm as it incurs an exponential blow-up in the
final dimension. They reduce the dimension by repeated application of the rounding step and the constructive version of the regularity lemma. Finally, it outputs a matrix $\hat{B}$ of rank $r^{\prime} d^{\prime \prime}$ where $r^{\prime} \geqslant r+1$ and $d^{\prime \prime} \leqslant r^{\prime}+1$.

## Our algorithm

One of the main features of our algorithm is that, we implement the rank increment step without using the second Wong sequence. At a high level, the rank increment step of our algorithm has a conceptual similarity with the proof idea used in [22]. If for a matrix tuple $p$ of dimension $d$, the rank of $T(p) \geqslant r d$, then we say that $\underline{p}$ is a witness of noncommutative rank $r$ of $T$. Using simple linear aĪgebraic ideas, we reduce the rank increment step to the polynomial identity testing of noncommutative ABPs. As we show in Lemma 18, Lemma 19, and Corollary 20 that to find a witness of a larger rank, it suffices to compute a matrix tuple where a noncommutative ABP does not evaluate to zero. This requires applying simple and well-known identity testing algorithms [23, 24]. This way one can compute a matrix tuple $p^{\prime}$ of dimension $d_{1}$ such that the rank of $T\left(\underline{p}^{\prime}\right)>r d_{1}$. Interestingly, there is no intermediate bit-complexity blow-up over $\mathbb{Q}$.

The rest of the algorithm follows the rounding and the blow-up control steps of [2] closely. In Lemma 13, we show how to embed $p^{\prime}$ in a division algebra $D$ of index $d_{1}$ to compute another tuple $\underline{p}^{\prime \prime}$, a witness of rank $r^{\prime}$ of $T$ where $r^{\prime} \geqslant r+1$. More precisely, the rank of $T\left(p^{\prime \prime}\right) \geqslant(r+1) d_{1}$. For the division algebra $D$, we use a well-known explicit construction of cyclic division algebra from [25].

We then use Lemma 21 to control the blow-up in the dimension and output the matrix tuple $\widehat{p}$ of dimension at most $d^{\prime}$ such that $\operatorname{rank}(T) \geqslant(r+1) d^{\prime}$ where $d^{\prime} \leqslant r^{\prime}+1$. Hence $\widehat{p}$ is a witness $\overline{\text { that }} \operatorname{ncrank}(T) \geqslant r+1$. We need to repeat the entire procedure for at most $s$ rounds where $s$ is the dimension of $T$.

We finally remark that unlike [2], the rank increment step does not need an explicit description of the coefficient matrices of $T$ if we settle with a quasipolynomial-time algorithm. Essentially, we can use the quasipolynomial-size explicit hitting set construction for the identity testing of noncommutative ABPs by Forbes and Shpilka [4] in place of the algorithm by Raz and Shplika [23]. We elaborate on this in Section 4. We believe that the connection with the noncommutative polynomial identity testing could be useful in understanding the black-box and the parallel complexity of NSINGULAR which are long-standing open problems [1, 19].

Organization. In Section 2, we collect background results from algebraic complexity theory and cyclic division algebras. We prove Theorem 1 in Section 3. The main contribution of the paper is in the implementation of the rank increment step which is presented in Subsection 3.2.1. The final algorithm is presented in Subsection 3.3. Section 4 contains some additional remarks related to the rank increment step in the black-box setting.

## 2 Background and Notation

Throughout the paper, we use $\mathbb{F}, F, K$ for fields. $\mathbf{M}_{m}(\mathbb{F})$ (resp. $\mathbf{M}_{m}(F), \mathbf{M}_{m}(K)$ ) is used for $m$ dimensional matrix algebra over $\mathbb{F}$ where $m$ is clear from the context. Similarly, we use $\mathbf{M}_{m}(\mathbb{F})^{n}$ (resp. $\left.\mathbf{M}_{m}(F)^{n}, \mathbf{M}_{m}(K)^{n}\right)$ to denote the set of $n$ tuples over $\mathbf{M}_{m}(\mathbb{F})\left(\right.$ resp. $\mathbf{M}_{m}(F), \mathbf{M}_{m}(K)$ ). $D$ is used to denote a division algebra. Let $\underline{x}$ be the set of variables $\left\{x_{1}, \ldots, x_{n}\right\}$. Sometimes we use $\underline{p}, \underline{q}$ to denote the matrix tuples in suitable matrix algebras. The free noncommutative ring of polynomials over a field $\mathbb{F}$ is denoted by $\mathbb{F}\langle\underline{x}\rangle$. The notation $A \otimes B$ is the usual tensor product of the matrices $A, B$.

### 2.1 Algebraic Complexity

Definition 3 (Algebraic Branching Program). An algebraic branching program (ABP) is a layered directed acyclic graph. The vertex set is partitioned into layers $0,1, \ldots, d$, with directed edges only between adjacent layers ( $i$ to $i+1$ ). There is a source vertex of in-degree 0 in the layer 0 , and one out-degree 0 sink vertex in layer $d$. Each edge is labeled by an affine $F$-linear form. The polynomial computed by the ABP is the sum over all source-to-sink directed paths of the ordered product of affine forms labeling the path edges.

The size of the ABP is defined as the total number of nodes and the width is the maximum number of nodes in a layer. An ABP can compute a commutative or a noncommutative polynomial. ABPs of width $w$ can also be seen as iterated matrix multiplication $\underline{c} \cdot M_{1} M_{2} \cdots M_{\ell} \cdot \underline{b}$, where $\underline{c}, \underline{b}$ are $1 \times w$ and $w \times 1$ vectors respectively and each $M_{i}$ is a $w \times w$ matrix, whose entries are affine linear forms over $\underline{x}$.

## Identity testing results

Given a noncommutative ABP, Raz and Shpilka have given a deterministic polynomial time algorithm to check whether the polynomial computed by the ABP is zero or not [23].

Theorem 4 (Raz-Shpilka [23]). Given a noncommutative ABP of width $w$ and $d$ many layers computing a polynomial $f \in \mathbb{F}\langle\underline{x}\rangle$, there is a deterministic $\operatorname{poly}(w, d, n)$ time algorithm to test whether $f \equiv 0$ or not.

In fact, the following corollary is standard by now. This was first formally observed in [24] using a minor adaptation of [23].
Corollary 5. Given a noncommutative $A B P$ of width $w$ and $d$ many layers computing a nonzero polynomial $f \in \mathbb{F}\langle\underline{x}\rangle$, there is a deterministic poly $(w, d, n)$ time algorithm which outputs a nonzero monomial $m$ in $f$. If $\mathbb{F}=\mathbb{Q}$, the bit-complexity of the algorithm is $\operatorname{poly}(w, d, n, b)$ where $b$ is the maximum bit-complexity of any coefficient in the input $A B P$.

Essentially, the algorithm of Raz and Shpilka maintains basis vectors (indexed by at most $w$ monomials) in each layer of the ABP using simple linear algebraic computations. The entries of the basis vectors are the coefficients of the indexing monomials in different nodes of the ABP along the width.

Given such a monomial $m=x_{i_{1}} x_{i_{2}} \ldots x_{i_{d}}$, [24] introduced a simple trick to produce a matrix tuple in $\mathbf{M}_{d+1}(\mathbb{F})^{n}$ on which $f$ evaluates to nonzero. To see that consider a $d+1$ state deterministic finite automaton $\mathcal{A}$ that accepts only the string $x_{i_{1}} x_{i_{2}} \ldots x_{i_{d}}$ over the alphabet $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. The transition matrix tuple $\left(M_{x_{1}}, \ldots, M_{x_{n}}\right)$ of $\mathcal{A}$ have the property that $f\left(M_{x_{1}}, \ldots, M_{x_{n}}\right) \neq 0$. More precisely, the automaton $\mathcal{A}$ is the following.


The transition matrices $M_{x_{j}}: 1 \leqslant j \leqslant n$ are $(d+1)$ dimensional $(0,1)$-matrices with the property that $M_{x_{j}}(\ell, \ell+1)=1$ if and only if $x_{j}$ is the edge label between $q_{\ell}$ and $q_{\ell+1}$ for $0 \leqslant \ell \leqslant d-1$. This we record as a corollary.
Corollary 6. Given a noncommutative $A B P$ of width $w$ and $d$ layers computing a nonzero polynomial $f \in$ $\mathbb{F}\langle\underline{x}\rangle$, there is a deterministic polynomial-time algorithm that can output a matrix tuple $\left(M_{1}, M_{2}, \ldots, M_{n}\right)$ of dimension at most $d+1$ such that $f\left(M_{1}, M_{2}, \ldots, M_{n}\right) \neq 0$.

### 2.2 Cyclic Division Algebras

A division algebra $D$ is an associative algebra over a (commutative) field $\mathbb{F}$ such that all nonzero elements in $D$ are units (they have a multiplicative inverse). In the context of this paper, we are interested in finite-dimensional division algebras. Specifically, we focus on cyclic division algebras and their construction [25, Chapter 5]. Let $F=\mathbb{Q}(z)$, where $z$ is a commuting indeterminate. Let $\omega$ be an $\ell^{t h}$ primitive root of unity. To be specific, let $\omega=e^{2 \pi i / \ell}$. Let $K=F(\omega)=\mathbb{Q}(\omega, z)$ be the cyclic Galois extension of $F$ obtained by adjoining $\omega$. The elements of $K$ are polynomials in $\omega$ (of degree at most $\ell-1$ ) with coefficients from $F$.

Define $\sigma: K \rightarrow K$ by letting $\sigma(\omega)=\omega^{k}$ for some $k$ relatively prime to $\ell$ and stipulating that $\sigma(a)=a$ for all $a \in F$. Then $\sigma$ is an automorphism of $K$ with $F$ as fixed field and it generates the Galois group $\operatorname{Gal}(K / F)$.

The division algebra $D=(K / F, \sigma, z)$ is defined using a new indeterminate $x$ as the $\ell$ dimensional vector space:

$$
D=K \oplus K x \oplus \cdots \oplus K x^{\ell-1},
$$

where the (noncommutative) multiplication for $D$ is defined by $x^{\ell}=z$ and $x b=\sigma(b) x$ for all $b \in K$. Then $D$ is a division algebra of dimension $\ell^{2}$ over $F$ [25, Theorem 14.9]. The index of $D$ is defined to be the square root of the dimension of $D$ over $F$. In our example, $D$ is of index $\ell$.

The elements of $D$ has matrix representation in $K^{\ell \times \ell}$ from its action on the basis $X=$ $\left\{1, x, \ldots, x^{\ell-1}\right\}$. I.e., for $a \in D$ and $x^{j} \in \mathcal{X}$, the $j^{t h}$ row of the matrix representation is obtained by writing $x^{j} a$ in the $\mathcal{X}$-basis.

For example, the matrix representation $M(x)$ of $x$ is:

$$
M(x)[i, j]= \begin{cases}1 & \text { if } j=i+1, i \leqslant \ell-1 \\ z & \text { if } i=\ell, j=1 \\ 0 & \text { otherwise } .\end{cases}
$$

$$
M(x)=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
z & 0 & \cdots & 0 & 0
\end{array}\right]
$$

For each $b \in K$ its matrix representation $M(b)$ is:

$$
\begin{gathered}
M(b)[i, j]= \begin{cases}b & \text { if } i=j=1 \\
\sigma^{i-1}(b) & \text { if } i=j, i \geqslant 2 \\
0 & \text { otherwise. }\end{cases} \\
M(b)=\left[\begin{array}{cccccc}
b & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma(b) & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma^{2}(b) & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma^{\ell-2}(b) & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma^{\ell-1}(b)
\end{array}\right]
\end{gathered}
$$

Remark 7. We note that $M(x)$ has a "circulant" matrix structure and $M(b)$ is a diagonal matrix. For a vector $v \in K^{\ell}$, it is convenient to write $\operatorname{circ}\left(v_{1}, v_{2}, \ldots, v_{\ell}\right)$ for the $\ell \times \ell$ matrix with $(i, i+1)^{\text {th }}$ entry $v_{i}$ for $i \leqslant \ell-1,(\ell, 1)^{t h}$ entry as $v_{\ell}$ and remaining entries zero. Thus, we have $M(x)=\operatorname{circ}(1,1, \ldots, 1, z)$. Similarly, we write $\operatorname{diag}\left(v_{1}, v_{2}, \ldots, v_{\ell}\right)$ for the diagonal matrix with entries $v_{i}$.

Fact 8. The F-algebra generated by $M(x)$ and $M(b), b \in K$ is an isomorphic copy of the cyclic division algebra in the matrix algebra $\mathbf{M}_{\ell}(K)$.

Proposition 9. For all $b \in K, \operatorname{circ}\left(b, \sigma(b), \ldots, z \sigma^{\ell-1}(b)\right)=M(b) \cdot M(x)$.
Define $C_{i, j}=M\left(\omega^{j-1}\right) \cdot M\left(x^{i-1}\right)$ for $1 \leqslant i, j \leqslant \ell$. Observe that, $\mathfrak{B}=\left\{C_{i j}, i, j \in[\ell]\right\}$ be a $F$-generating set for the division algebra $D$. The following proposition is a standard fact.

Proposition 10. [25, Section 14(14.13)] Then $K$ linear span of $\mathfrak{B}$ is the entire matrix algebra $\mathbf{M}_{\ell}(K)$.

## 3 Noncommutative Rank Computation

In this section, we present the proof of Theorem 1 For the sake of the reader, let us first recall the definition of the inner rank, the blow-up rank of a linear matrix, and their equivalence from Section 1.1 .

The noncommutative rank (ncrank) or the inner rank of an $s \times s$ linear matrix $T$ over the noncommuting variables $x_{1}, x_{2}, \ldots, x_{n}$ is the minimum $r$ such that $T=P \cdot Q$ where $P$ is an $s \times r$ matrix and $Q$ is an $r \times s$ matrix with entries linear in $x_{1}, \ldots, x_{n}$ [12].

Let $T$ be an $s \times s$ matrix whose entries are linear forms over $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. We can write $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$ where $A_{0}, A_{1}, \ldots, A_{n}$ are the coefficient matrices. Given such a matrix $T$ over the variables $x_{1}, \ldots, x_{n}$ and $d \in \mathbb{N}$, let

$$
T^{\{d\}}=\left\{T(\underline{p}) \mid \underline{p} \in \mathbf{M}_{d}(\mathbb{F})^{n}\right\} .
$$

Here $T(\underline{p})=A_{0} \otimes I_{d}+\sum_{i=1}^{n} A_{i} \otimes p_{i}$. Define $\operatorname{rank}\left(T^{\{d\}}\right)=\max _{\underline{p}}\{\operatorname{rank}(T(\underline{p}))\}$. The regularity lemma [2, 20] shows that $\operatorname{rank}\left(T^{\{d\}}\right)$ is always a multiple of $d$. In Section 3.1, we discuss a constructive version of this lemma.

Definition 11. The blow-up rank of the matrix $T$ is defined as

$$
\operatorname{ncrank}^{*}(T)=\lim _{d \rightarrow \infty} \frac{\operatorname{rank}\left(T^{\{d\}}\right)}{d} .
$$

Using the regularity lemma and the weakly increasing property of the sequence $\left(\frac{\operatorname{rank}\left(T^{\{d\}}\right)}{d}\right)_{d \geqslant 1}$, it is shown that the limit exists [26, Chapter 4]. For any linear matrix $T$, it is known that ncrank $(T)=$ $n^{n} \mathrm{crank}^{*}(T)$ [2]. Henceforth, we work with the blow-up rank of $T$ but continue to denote it by $\operatorname{ncrank}(T)$ for notational simplicity. The blow-up rank motivates us to define the following notion of a witness.

Definition 12 (Witness of rank $r$ ). Let $A_{0}, A_{1}, \ldots, A_{n} \in \mathbf{M}_{s}(\mathbb{F})$ and $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$. We say that $\underline{p}=\left(p_{1}, \ldots, p_{n}\right) \in \mathbf{M}_{d}(\mathbb{F})^{n}$ for some $d$ is a witness of noncommutative rank $r$ of $T$, if $\operatorname{rank}(T(\underline{p})) \geqslant r d$.

### 3.1 Constructive Regularity Lemma

Suppose that for a linear matrix $T$, we already have a matrix tuple $q$ of dimension $d$, a witness of $\operatorname{rank} r$ of $T$ such that $\operatorname{rank}(T(q))>r d$. Then the constructive regularity lemma offers a simple and general procedure to get a $\bar{d} \times d$ witness of rank $r+1$ for $T$ [2]. We state essentially the same proof as described in [2]. But for clarity and simplicity, we use the explicit cyclic division algebra construction described in Section [2.2. Following Section[2.2, the field $F=\mathbb{Q}(z)$ and $K=F(\omega)$.

Lemma 13. [2] For any $s \times s$ matrix $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$, and a matrix tuple $\underline{q}=\left(q_{1}, \ldots, q_{n}\right) \in \mathbf{M}_{d}(K)^{n}$ such that $\operatorname{rank}(T(\underline{q}))>r d$, there exists a deterministic $\operatorname{poly}(n, s, d)$-time algorithm that returns another matrix substitution $\underline{q}^{\prime}=\left(q_{1}^{\prime}, \ldots, q_{n}^{\prime}\right) \in \mathbf{M}_{d}(K)^{n}$ such that $\operatorname{rank}\left(T\left(q^{\prime}\right)\right) \geqslant(r+1) d$.

Proof. Let $D=(K / F, \sigma, z)$ be the cyclic division algebra described in Section [2.2. Recall that $F=\mathbb{Q}(z)$ and $K=F(\omega)$ and $\mathfrak{B}=\left\{C_{i, j}: i, j \in[d]\right\}$ is a $F$-generating set of $D$.

1. Using Proposition 10, we can express $q_{k}=\sum_{i, j} \lambda_{i, j, k} C_{i, j}$ where $\lambda_{i, j, k}: 1 \leqslant k \leqslant n$ are unknown variables which take values in $K$. Using linear algebra we determine the values $\lambda_{i, j, k}^{0}: 1 \leqslant i, j \leqslant \ell, 1 \leqslant k \leqslant n$ for the unknowns in $K$.
2. Now the goal is to compute a $d \times d$ tuple $\underline{q}^{\prime}=\left(q_{1}^{\prime}, \ldots, q_{n}^{\prime}\right)$ such that $q_{k}^{\prime}=\sum_{i, j} \mu_{i, j, k}^{0} C_{i, j}$ where $\mu_{i, j, k}^{0} \in F$ and $\operatorname{rank}\left(T\left(q^{\prime}\right)\right) \geqslant(r+1) d$. We briefly describe the procedure outlined in [2]. Write $\tilde{q}_{1}=\mu_{1,1,1} C_{1,1,1}+\sum_{(i, j) \neq(1,1)} \lambda_{i, j, 1}^{0} C_{i, j}$ where $\mu_{1,1,1}$ is a variable. There will be a sub-matrix of size $>r d$ whose minor is non-zero, under the current substitution $\left(\tilde{q}_{1}, q_{2}, \ldots, q_{n}\right)$. Since the determinant of that sub-matrix is a univariate polynomial in $\mu_{1,1,1}$ and degree $\operatorname{poly}(r, d)$, we can easily fix the value of $\mu_{1,1,1}$ from $\mathbb{Q}$ such that the minor remains nonzero. Repeating the procedure, we can compute the desired tuple $\underline{q}^{\prime}$. Since $\underline{q}^{\prime}$ is a tuple over the division algebra, $\operatorname{rank}\left(T\left(\underline{q}^{\prime}\right)\right) \geqslant(r+1) d$.

Remark 14. The last line of the above proof is easy to see. The matrix $T\left(q^{\prime}\right)$ can be viewed as a $s \times s$ block-matrix of $d$-dimensional blocks, and each such block is an element in $D$. Since Gaussian elimination is supported over division algebras, up to elementary row and column operations, we can transform $T\left(q^{\prime}\right)$ as:

$$
\left(\begin{array}{c|c}
I & 0 \\
\hline 0 & 0
\end{array}\right)
$$

where $I$ is an identity matrix which has at least $r+1$ blocks of identity matrices $I_{d}$ on its diagonal. Hence $\operatorname{rank}\left(T\left(q^{\prime}\right)\right) \geqslant(r+1) d$.

### 3.2 The Plan of the Algorithm

Following [2], we first give a simple template.

## Algorithm Template

Input: $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$ where $A_{0}, A_{1}, \ldots, A_{n} \in \mathbf{M}_{s}(\mathbb{Q})$.
Output: The noncommutative rank of $T$.

The algorithm gradually constructs a witness at every stage. Suppose we already have a witness of rank $r$ for $T$.

1. Is $r$ the maximum rank?
2. If yes, output $r$ to be the noncommutative rank of $T$.
3. Otherwise, find a witness of rank at least $r+1$ and go to Step 1.

We now discuss each step in detail.

### 3.2.1 Rank Increment Step

For an $s \times s$ linear matrix $T(\underline{x})=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$ and $d \in \mathbb{N}$, define

$$
T_{d}(Z)=A_{0} \otimes I_{d}+\sum_{i=1}^{n} A_{i} \otimes Z_{i}
$$

where $Z_{i}=\left(z_{j k}^{(i)}\right)_{1 \leqslant j, k \leqslant d}$ is a $d \times d$ generic matrix with noncommutative indeterminates. In other words, $Z=\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right)$ is the substitution used for the variables $x_{1}, x_{2}, \ldots, x_{n}$ in $T$. Now $T_{d}(Z)$ is a linear matrix of dimension sd over the variables $\left\{z_{j k}^{(i)}\right\}_{1 \leqslant j, k \leqslant d, 1 \leqslant i \leqslant n}$.

Remark 15. It is immediate to see that any $d \times d$ matrix shift $T_{d}\left(Z_{1}+p_{1}, Z_{2}+p_{2}, \ldots, Z_{n}+p_{n}\right)$ is indeed a scalar shift for the variables $\left\{z_{j k}^{(i)}\right\}_{1 \leqslant j, k \leqslant d, 1 \leqslant i \leqslant n}$ in the matrix $T_{d}$.

Lemma 16. $\operatorname{ncrank}\left(T_{d}\right)=d \cdot \operatorname{ncrank}(T)$.
Proof. Let $\operatorname{ncrank}(T)=r$. Therefore, for every sufficiently large $d^{\prime \prime}$, the maximum rank obtained by evaluating $T$ over all the $d^{\prime \prime} \times d^{\prime \prime}$ matrix tuple is $r d^{\prime \prime}$. Consider $d^{\prime \prime}=d d^{\prime}$, a multiple of $d$ and let $\underline{q}=\left(q_{1}, \ldots, q_{n}\right) \in \mathbf{M}_{d^{\prime \prime}}(K)^{n}$ be a matrix tuple such that $\operatorname{rank}(T(\underline{q}))=r d d^{\prime}$. Let

$$
\underline{p}=\left(p_{11}^{(1)}, \ldots, p_{d d}^{(1)}, \ldots, p_{11}^{(n)}, \ldots, p_{d d}^{(n)}\right)
$$

be the matrix tuple in $\mathbf{M}_{d^{\prime}}(K)^{n d^{2}}$ such that each $q_{i}=\left(p_{j k}^{(i)}\right)_{1 \leqslant j, k \leqslant d}$, i.e. we think of $q_{i}$ as $d \times d$ block matrix where the $(j, k)^{t h}$ block is $p_{j k}^{(i)}$. Notice that $T_{d}(\underline{p})=A_{0} \otimes I_{d d^{\prime}}+\sum_{i=1}^{n} A_{i} \otimes q_{i}=T(\underline{q})$. Notice that, the matrix $q_{i}$ is substituted for the variable $x_{i}$ in $T$. Therefore, $\operatorname{rank}(T(\underline{q}))=\operatorname{rank}\left(T_{d}(\underline{p})\right)$ and $\operatorname{ncrank}\left(T_{d}\right) \geqslant r d$.

For the other direction, as $\operatorname{ncrank}(T)=r$, we can write $T=P \cdot Q$ where $P, Q$ are $s \times r$ and $r \times s$ matrices respectively with linear entries. We can now define an $s d \times r d$ matrix $P^{\prime}(Z)$ by substituting each $x_{i}$ by $Z_{i}$ in the matrix $P(\underline{x})$. Similarly, we can define a $r d \times s d$ matrix $Q^{\prime}(Z)$ from $Q(\underline{x})$. Notice that, $T_{d}=P^{\prime} \cdot Q^{\prime}$. Therefore, $\operatorname{ncrank}\left(T_{d}\right) \leqslant r d$. Hence, the lemma follows.

Suppose, we have already computed a witness of noncommutative rank $r$ of $T$, namely $p=$ $\left(p_{1}, \ldots, p_{n}\right) \in \mathbf{M}_{d}(K)^{n}$ (by construction, we will ensure that $d \leqslant r+1$ ). We now check whether $\operatorname{ncrank}(T)>r$ or not.

$$
\text { Observe that, } T_{d}\left(Z_{1}+p_{1}, \ldots, Z_{n}+p_{n}\right)=U\left(\begin{array}{c|c}
I_{r d}-L & A \\
\hline B & C
\end{array}\right) V
$$

for invertible transformations $U, V$ in $\mathbf{M}_{r d}(K)$. In fact, using further invertible transformations $U^{\prime}, V^{\prime}$ over the free skew field $K \nless Z \gtrdot$ we can write

$$
\begin{aligned}
& T_{d}\left(Z_{1}+p_{1}, \ldots, Z_{n}+p_{n}\right)=U U^{\prime}\left(\begin{array}{c|c}
I_{r d}-L & 0 \\
\hline 0 & C-B\left(I_{r d}-L\right)^{-1} A
\end{array}\right) V^{\prime} V . \\
& \text { Here, } U^{\prime}=\left(\begin{array}{c|c}
I_{r d} & 0 \\
\hline B\left(I_{r d}-L\right)^{-1} & I_{(s-r) d}
\end{array}\right), \quad V^{\prime}=\left(\begin{array}{c|c}
I_{r d} & \left(I_{r d}-L\right)^{-1} A \\
\hline 0 & I_{(s-r) d}
\end{array}\right) .
\end{aligned}
$$

Let $\widetilde{T_{d}}=C-B\left(I_{r d}-L\right)^{-1} A$. Notice that the $(i, j)^{t h}$ entry of $\widetilde{T}_{d}$ is given by $\widetilde{\left(T_{d}\right)}{ }_{i j}=C_{i j}-B_{i}\left(I_{r d}-L\right)^{-1} A_{j}$ where $B_{i}$ is the $i^{\text {th }}$ row vector of $B$ and $A_{j}$ is the $j^{\text {th }}$ column vector of $A$.

Lemma 17. $\operatorname{ncrank}(T)>r$ if and only if $\widetilde{\left(T_{d}\right)}{ }_{i j} \neq 0$ for some choice of $i, j$.
Proof. Let $\operatorname{ncrank}(T)>r$. Then by Lemma 16, $\operatorname{ncrank}\left(T_{d}\right)>r d$. The noncommutative rank of a linear matrix is invariant under a scalar shift 2 , hence $\operatorname{ncrank}\left(T_{d}\left(Z_{1}+p_{1}, \ldots, Z_{n}+p_{n}\right)\right)=\operatorname{ncrank}\left(T_{d}\right)>$ $r d$. However, if $C-B\left(I_{r d}-L\right)^{-1} A$ is a zero matrix, this is impossible.

Conversely if ${\widetilde{\left(T_{d}\right)}}_{i j}=C_{i j}-B_{i}\left(I_{r d}-L\right)^{-1} A_{j}$ is nonzero for some indices $i, j$, we can find matrix substitutions $\tilde{p}_{\ell_{1} \ell_{2}}^{(k)}$ of dimension $d^{\prime}$ for the variables $\left\{z_{\ell_{1} \ell_{2}}^{(k)}\right\}_{1 \leqslant \ell_{1}, \ell_{2} \leqslant d, 1 \leqslant k \leqslant n}$, such that the rank of $T_{d}\left(Z_{1}+p_{1}, \ldots, Z_{n}+p_{n}\right)$ on that substitution is more than $r d d^{\prime}$. Therefore, $\operatorname{ncrank}\left(T_{d}\left(Z_{1}+p_{1}, \ldots, Z_{n}+\right.\right.$ $\left.\left.p_{n}\right)\right)>r d$. Hence $\operatorname{ncrank}\left(T_{d}\right)>r d$. By Lemma 16, we get that $\operatorname{ncrank}(T)>r$.

The next lemma says that the infinite series $\widetilde{\left(T_{d}\right)_{i j}} \neq 0$ is equivalent in saying that a suitably truncated polynomial $\widetilde{P}_{i j}$ obtained from ${\left.\widetilde{T_{d}}\right)_{i j} \text { is nonzero. The proof of the lemma is fairly standard [27, }}_{\text {[ }}$, Corollary 8.3, Page 145]. However, we present a self-contained proof of it.

Lemma 18. ${\widetilde{\left(T_{d}\right)}}_{i j} \neq 0$ if and only if $\widetilde{P}_{i j}=C_{i j}-B_{i}\left(\sum_{k \leqslant r d-1} L^{k}\right) A_{j} \neq 0$.
Proof. We first notice that, ${\widetilde{\left(T_{d}\right)}}_{i j}$ is zero in the free skew field over the $Z$-variables if and only if the formal power series $C_{i j}-B_{i}\left(\sum_{k \geqslant 0} L^{k}\right) A_{j}$ is zero. Therefore, if $\widetilde{\left(T_{d}\right)_{i j}}=0$, then obviously $\widetilde{P}_{i j}=0$ as the power series is zero.

Now suppose $\widetilde{P}_{i j}=0$. Note the terms in $C_{i j}$ are linear forms and the degree of any term in $B_{i}\left(\sum_{k \geqslant 0} L^{k}\right) A_{j}$ is at least two. So $C_{i j}$ must be zero. For simplicity, identify the Z-variables with $z_{1}, z_{2}, \ldots, z_{N}$ where $N=n d^{2}$. Write the row and column vectors $B_{i}$ and $A_{j}$ as $B_{i}=\sum_{\ell} B_{i, \ell} z_{\ell}, A_{j}=$ $\sum_{\ell} A_{j, \ell} z_{\ell}$. Similarly, write $L=\sum_{\ell} L_{\ell} z_{\ell}$. Let, if possible, $B_{i} L^{r d} A_{j}$ contributes a nonzero monomial (word) $w=z_{i_{1}} z_{i_{2}} \ldots z_{i_{r d+2}}$. Clearly the coefficient of $w$ is $B_{i, i_{1}} L_{i_{2}} \ldots L_{i_{r d+1}} A_{j, i_{r d+2}}$. Look at the vectors $v_{1}=B_{i, i_{1}}, v_{2}=B_{i, i_{1}} L_{i_{2}}, \ldots, v_{r d+1}=B_{i, i_{1}} L_{i_{2}} \ldots L_{i_{r d+1}}$ corresponding to the prefix words $w_{1}=$ $z_{i_{1}}, w_{2}=z_{i_{1}} z_{i_{2}}, \ldots, w_{r d+1}=z_{i_{1}} \ldots z_{i_{r d+1}}$. They can not be all linearly independent since they are $r d$-dimensional vectors. Hence there exists scalars $\lambda_{1}, \ldots, \lambda_{r d+1}$ such that $\lambda_{1} v_{1}+\ldots \lambda_{r d+1} v_{r d+1}=0$. However, $v_{r d+1} A_{j, i_{r d+2}} \neq 0$ by the assumption. Hence there exists at least one vector $v_{\ell}: 1 \leqslant \ell \leqslant r d$ such that $v_{\ell} A_{j, i_{r d+2}} \neq 0$. This means that the coefficient of the word $w_{\ell} z_{i_{r d+2}}$ of length at most $r d+1$ is nonzero in $\widetilde{P}_{i j}$, which is not possible.

We now repeat the same argument to show the infinite series $B_{i}\left(\sum_{k \geqslant 0} L^{k}\right) A_{j}=0$.
${ }^{2}$ Consider a linear matrix $L$ that achieves the maximum rank for a matrix substitution $\underline{q}$ of some dimension $d$. Then, for any scalar shift $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, the shifted linear matrix $L(\underline{x}+\underline{\alpha})$ achieves the same rank on the matrix tuple $\underline{q}-\underline{\alpha} \otimes I_{d}$.

Next, we apply Corollary 5 and Corollary 6 to output a matrix tuple efficiently on which $\widetilde{\left(T_{d}\right)_{i j}}$ evaluates to nonzero and $I_{r d}-L$ evaluates to a full rank matrix.

Lemma 19. There is a deterministic poly $(n, r, d)$-time algorithm that can output a matrix tuple $\underline{q}$ of dimension at most $d^{\prime}=2 r d$ for the $Z$ variables such that $I_{r d d^{\prime}}-L(\underline{q})$ is invertible and $\widetilde{\left(T_{d}\right)_{i j}}(\underline{q}) \neq 0$.

Proof. Notice that $\widetilde{P}_{i j}$ is an $\operatorname{ABP}$ of $\operatorname{size} \operatorname{poly}(r, d)$ and the number of layers is at most $r d+1$. Applying Corollary 6 , we get a matrix tuple of dimension at most $r d+2$ such that $\widetilde{P}_{i j}$ evaluates on it to nonzero. By simple padding, we can get a matrix tuple $q^{\prime}$ of dimension $d^{\prime}=2 r d$ such that $\widetilde{P}_{i j}\left(\underline{q}^{\prime}\right) \neq 0$. Since $\underline{q}^{\prime}$ is a substitution for the $Z$ variables $\left\{z_{\ell_{1} \ell_{2}}^{(k)}\right\}$ where $1 \leqslant k \leqslant n, 1 \leqslant \ell_{1}, \ell_{2} \leqslant d$, we write $q^{\prime}=\left(q_{11}^{(1)}, \ldots, q_{d d}^{\prime(1)}, \ldots, q_{11}^{\prime(n)}, \ldots, q_{d d}^{\prime(n)}\right)$ for more clarity. Here each $q_{\ell_{1} \ell_{2}}^{\prime(k)}$ is a $d^{\prime}$ dimensional matrix.

Consider a commutative variable $t$ and the scaled matrix tuple $t \underline{q}^{\prime}$. It is easy to see that the infinite series $C_{i j}-B_{i}\left(I_{r d d^{\prime}}-L\left(t \underline{q}^{\prime}\right)\right)^{-1} A_{j}$ is nonzero since the $k^{t h}$ homogeneous part $t^{k} B_{i} L^{k}\left(q^{\prime}\right) A_{j}$ will not mix with other homogeneous components.

However this also has a rational representation $\widetilde{\left(T_{d}\right)_{i j}}\left(t q^{\prime}\right)=\gamma_{1}(t) / \gamma_{2}(t)$ where $t$-degrees of the polynomials $\gamma_{1}(t), \gamma_{2}(t)$ are bounded by $r d d^{\prime}$. Moreover $I_{r d d^{\prime}}-L\left(t q^{\prime}\right)$ is an invertible matrix and the degree of $\operatorname{det}\left(I_{r d d^{\prime}}-L\left(t \underline{q}^{\prime}\right)\right)$ is bounded by $r d d^{\prime}$ over the variable $t$. Simply by varying the variable $t$ over a suitable large set $\Gamma$ of size $O(r d)$, we can fix a value for $t=t_{0}$ such that $\widetilde{\left(T_{d}\right)} i_{i j}\left(t_{0} \underline{q}^{\prime}\right) \neq 0$ and $I_{r d d^{\prime}}-L\left(t_{0} \underline{q}^{\prime}\right)$ is of rank $r d d^{\prime}$. Define $\underline{q}=t_{0} \underline{q}^{\prime}$.

Following is an immediate corollary.
Corollary 20. Suppose Lemma 19 outputs a matrix tuple $\underline{q}$. We can compute another matrix tuple $\underline{p^{\prime}}$ of dimension $d d^{\prime}$ which is a witness of $\operatorname{ncrank}(T)>r$.

Proof. Define the matrix tuple $\underline{q}^{\prime \prime}=\left(q_{11}^{\prime \prime(1)}, \ldots, q_{d d}^{\prime \prime(1)}, \ldots, q_{11}^{\prime \prime(n)}, \ldots, q_{d d}^{\prime \prime(n)}\right)$ where $q_{\ell_{1} \ell_{2}}^{\prime \prime(k)}=q_{\ell_{1} \ell_{2}}^{(k)}+p_{\ell_{1} \ell_{2}}^{(k)} \otimes$ $I_{d^{\prime}}$ is a $d^{\prime}$ dimensional matrix tuple for $1 \leqslant k \leqslant n, 1 \leqslant \ell_{1}, \ell_{2} \leqslant d$.

Lemma 19 shows that the rank of $T_{d}$ evaluated on the matrix tuple $q^{\prime \prime}$ is more than $r d d^{\prime}$. This is same as saying that $T_{d}(Z)$ is of rank more than $r d d^{\prime}$ when the variable $z_{\ell_{1}, \ell_{2}}^{k}: 1 \leqslant k \leqslant n, 1 \leqslant \ell_{1}, \ell_{2} \leqslant d$ is substituted by $q_{\ell_{1} 1_{2}}^{\prime \prime(k)}$. Hence $\operatorname{ncrank}\left(T_{d}\right)>r d$. By Lemma 16, we know that $\operatorname{ncrank}(T)>r$. Moreover, we obtain a matrix tuple $\underline{p}^{\prime}=\left(p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{n}^{\prime}\right)$ which is a witness of $\operatorname{ncrank}(T)>r$, where $p_{k}^{\prime}=\left(q_{\ell_{1} \ell_{2}}^{\prime \prime(k)}\right)_{1 \leqslant \ell_{1}, \ell_{2} \leqslant d}: 1 \leqslant k \leqslant n$. Notice that $\underline{p}^{\prime}$ is the substitution for the $\underline{x}$ variables.

### 3.2.2 Rounding and Blow-up Control

Next, we apply Lemma 13 which gives a rounding procedure to get a matrix tuple of dimension $d_{1}=d d^{\prime}$ to witness that $\operatorname{ncrank}(T)=r^{\prime}$ where $r^{\prime} \geqslant r+1$. Call that new matrix tuple as $\underline{p}^{\prime \prime}$.

However, we can not afford to have such a dimension blow-up for the witness matrix tuple in every step of the iteration as it incurs an exponential blow-up in the dimension of the final witness. To control that, we use a simple trick from [2] which we describe for the sake of completeness.

Lemma 21. Consider an $s \times$ s linear matrix $T$ and a matrix tuple $p^{\prime \prime}$ in $\mathbf{M}_{d_{1}}(K)^{n}$ such that $p^{\prime \prime}$ is a witness of rank $r^{\prime}$ of $T$. We can efficiently compute another matrix tuple $\widehat{\widehat{p}}$ of dimension at most $r^{\prime}+\overline{1}$ such that $\underline{\hat{p}}$ is also a witness of rank $r^{\prime}$ of $T$.

Proof. Consider a sub-matrix $A$ in $T\left(p^{\prime \prime}\right)$ such that $\operatorname{rank}(A)$ is at least $r^{\prime} d_{1}$. From each matrix in the tuple $p^{\prime \prime}$, remove the last row and the column to get another tuple $\tilde{p}$. We claim that the corresponding sub-matrix $A^{\prime}$ in $T(\widetilde{p})$ is of rank $>\left(r^{\prime}-1\right)\left(d_{1}-1\right)$ as long as $\bar{d}_{1}>r^{\prime}+1$. Otherwise, $\operatorname{rank}(A) \leqslant \operatorname{rank}\left(A^{\prime}\right)+2 r^{\prime} \leqslant\left(r^{\prime}-1\right)\left(d_{1}-1\right)+2 r^{\prime}=r^{\prime} d_{1}-d_{1}+r^{\prime}+1<r^{\prime} d_{1}$. Now we can use the constructive regularity lemma (Lemma 13) on the tuple $\widetilde{p}$ to obtain another witness of dimension $d_{1}-1$ which is a witness of rank $r^{\prime}$ of $T$. Applying the procedure repeatedly, we can control the blow-up in the dimension within $r^{\prime}+1$ and get the witness tuple $\widehat{\widehat{p}}$.

### 3.3 The Final Algorithm

We now summarize our overall strategy.

## Algorithm

Input: $T=A_{0}+\sum_{i=1}^{n} A_{i} x_{i}$ where $A_{0}, A_{1}, \ldots, A_{n} \in \mathbf{M}_{s}(\mathbb{Q})$.
Output: The noncommutative rank of $T$.
The algorithm gradually increases the rank and finds a witness for it. Suppose at any intermediate stage, we already have a matrix tuple $\underline{p}$ in $\mathbf{M}_{d}(K)^{n}$, a witness of rank $r$ of $T$.

1. (Is $r$ the maximum rank?) Use Theorem 4 to check whether the polynomial $\widetilde{P}_{i j} \neq 0$ (as defined in Lemma 18) for some choice of $i, j$.
2. If "NO", output $r$ to be the noncommutative rank of $T$.
3. (Otherwise, construct a witness of rank $r+1$ and repeat Step 1) We implement the following steps to construct a rank $(r+1)$-witness:
(a) [Rank increment step] Apply Corollary 20 to find a $d_{1} \times d_{1}$ matrix substitution $\underline{p}^{\prime}=\left(p_{1}^{\prime}, \ldots, p_{n}^{\prime}\right)$ such that $\operatorname{rank}\left(T\left(\underline{p}^{\prime}\right)\right)>r d_{1}$ where $d_{1}=2 r d^{2}$.
(b) [Rounding using the regularity lemma] Apply Lemma 13] to find another $d_{1} \times d_{1}$ matrix substitution $\left(p_{1}^{\prime \prime}, \ldots, p_{n}^{\prime \prime}\right)$ such that the rank of $T$ evaluated at $\left(p_{1}^{\prime \prime}, \ldots, p_{n}^{\prime \prime}\right)$ is $r^{\prime} d_{1}$ where $r^{\prime} \geqslant r+1$.
(c) [Reducing the witness size] Apply Lemma 21 to find a matrix substitution $\widehat{\hat{p}}=$ $\left(\widehat{p}_{1}, \ldots, \widehat{p}_{n}\right)$ of dimension $d^{\prime} \leqslant r^{\prime}+1$, such that the rank of $T$ evaluated at $\underline{\hat{p}}$ is $\geqslant \bar{r}^{\prime} d^{\prime}$.

## Analysis

Since the noncommutative rank of $T$ is at most $s$, the algorithm iterates at most $s$ steps. Lemma 18 , Theotem 4 , and Lemma 19 guarantee that Step 1 and Step 3(a) can be done in poly $(n, r, d)$ steps. Step 3(b) and 3(c) require straightforward linear algebraic computations discussed in Section 3.2.2 which can be performed in $\operatorname{poly}(n, d, r)$ time. Since $d \leqslant s+1$ throughout the process, the run time is bounded by poly $(n, s)$. Understanding the bit-complexity of the algorithm is very simple. Let the witness of rank $r$ has bit-complexity $b$. In the rank increment step the matrix constructed in Corollary 6 has only 0,1 entries and the parameter $t_{0}$ is of poly $(s, d)$. So the bit-complexity after step 3(a) can change to $b+\log (s d)$ at most. Step 3(b) is a simple linear algebraic step that can at most incur bit-complexity by an additive factor $\operatorname{poly}(s, d)$. Therefore, the overall bit-complexity of the algorithm is poly(s).

## 4 Further Remarks

It is well-known that testing whether a bipartite graph has a perfect matching can be reduced to NSINGULAR using Hall's theorem [19]. The bipartite matching problem is in (black-box) quasiNC via a hitting set construction [28]. In contrast, designing an efficient black-box algorithm (or even a parallel algorithm) for NSINGULAR is wide open [19].

Interestingly, the algorithm presented in this paper can be adapted to get a black-box solution for the following problem.
Given a 4-tuple $\langle s, r, d, n\rangle \in \mathbb{N}^{4}$, construct efficiently a universal set $\mathcal{H}_{s, r, n, d} \subseteq \mathbf{M}_{k r d}(\mathbb{Q})^{n}$ (for some integer $k$ ) of sub-exponential size such that the following is true:

Consider any $\mathbb{Q}$-linear matrix $T$ defined over $x_{1}, x_{2}, \ldots, x_{n}$ and a tuple $p \in \mathbf{M}_{d}(\mathbb{Q})^{n}$ such that $\operatorname{rank}(T(p)) \geqslant r d$. Then, $\operatorname{ncrank}(T)>r$ if and only if there exists a tuple $\underline{q} \in \mathcal{H}_{s, r, n, d}$ such that $\operatorname{rank}\left(T\left(\underline{\bar{q}}+\underline{p} \otimes I_{k r}\right)\right)>r d$.

The connection with the noncommutative ABP identity testing enables us to construct such a set of quasipolynomial-size. We need to use the quasipolynomial-size hitting set construction for noncommutative ABPs by Forbes and Shpilka [4]. More precisely, in the proof of Lemma 19 we need to use the hitting set of [4] in place of Corollary 20. The rest of the arguments remain exactly the same. The size of the set $\mathcal{H}_{s, r, n, d}$ will be $(\operatorname{srnd})^{O(\log r d)}$ and the parameter $k$ will be $2 r d$. This step is black-box in the sense that it does not use the explicit description of the coefficient matrices $A_{0}, A_{1}, A_{2}, \ldots, A_{n}$. It is unclear how to solve this problem using the second Wong sequence-based techniques [21, 2].

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[^1]:    ${ }^{1}$ We say $U \leqslant \mathbb{F}^{n}$ is a $c$-shrunk subspace of a matrix space $\mathcal{B}$ if there exists $W \leqslant \mathbb{F}^{n}$ such that $\operatorname{dim}(W) \leqslant \operatorname{dim}(U)-c$ and for all $B \in \mathcal{B}$, the set $\{B u: u \in U\} \leqslant W$.

